HOMOCLINIC ORBITS IN SADDLE-CENTER REVERSIBLE HAMILTONIAN SYSTEMS

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Abstract

We study the existence of homoclic solutions for reversible Hamiltonian systems taking the family of differential equations $u^{iv} + au'' - u + f(u, b) = 0$ as a model. Here f is an analytic function and a, b real parameters. These equations are important in several physical situations such as solitons and in the existence of "finite energy" stationary states of partial differential equations. We reduce the problem of computing these orbits to that of finding the intersection of the unstable manifold with a suitable set and then apply it to concrete situations. No assumptions of any kind of discrete symmetry is made and the analysis here developed can be successfully employed in situations where standard methods fail.

Key words: reversible hamiltonian systems, homoclinic orbits, saddle-center singularity

1 Introduction

Homoclinic orbits have attracted the attention of several authors due to their important role as a mechanism leading to chaotic dynamics. This phenomenon

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was first analyzed by Poincaré, and the study of the dynamics in the neighborhood of homoclinic orbits was further developed by Birkhoff, Smale and Silnikov (see [1]).

We say that an orbit ϕ is homoclinic to a certain critical set p of a dynamical system (it could be an equilibrium point or a periodic orbit) if the orbit is bi-asymptotic to this set, that is, $\lim_{t\pm\infty} \phi(t) = p$. In this work we concentrate on the problem of existence of homoclinic orbits to saddle-center equilibrium points, in the context of reversible Hamiltonian systems (see below). Such issue and its implications are discussed in [2], [3], [4], [5] and [6]. The problem of finding homoclinic solutions is related to the existence of solitary waves, specially in the presence of surface tension [7], elastic structures [8],[9],[10],[11] and spatial patterns in phase transition [12] and [13]. Also, homoclinic solutions are important to proving the existence of stationary finite energy states in partial differential equations [14].

In general a Hamiltonian system with a saddle-center equilibrium r does not possess homoclinic orbits to r. In order for this orbit to exist it is necessary and sufficient that the 1-dimensional stable and unstable manifolds to r, defined on the same 3-dimensional energy surface, intersect. As discussed in Section 2, the reversibility of the system will alter completely this situation, and more interesting cases arise. We use as a model the equation presented in the abstract, but our results are more general. In Section 3 we apply the ideas developed here to a specific equation by doing the necessary analytical and numerical work considerations. In Section 4 we discuss additional applications and report on future simulations to explore in more detail the rich consequences of the method here presented

2 DISCUSSION OF THE METHOD

Consider a two degrees of freedom family of Hamiltonian systems depending on two real parameters a and b, $(M, \omega, H(a,b))$, where M is a four-dimensional C^∞ manifold, ω a symplectic form (closed, non degenerate 2-form over M), and $H(a,b) : M \to R$ is the Hamiltonian function. Let $X(M)$ be the set of C^∞ vector fields over M. Given H there exists a vector field $X_H \in X(M)$ defined by

$$\omega(X_H, Y) = dH(Y), \text{ for all } Y \in X(M).$$

The flow $\Psi : R \times M \to M$ is defined as $\frac{\partial \Psi(.,x)}{\partial t} = X_H(\Psi(.,x))$ and, in symplectic coordinates (q_1, q_2, p_1, p_2), the solution is given by the Hamilton equations $\dot{q}_i = \frac{\partial H}{\partial p_i}$ and $\dot{p}_i = -\frac{\partial H}{\partial q_i}$.

Our Hypotheses are:

H1. $(M, \omega, H(a, b))$ has a saddle-center equilibrium $r = \overrightarrow{0}$, that is, the linearized field X_H at r has a pair of real eigenvalues and a pair of pure imaginary eigenvalues (non hyperbolic equilibrium point).

H2. $(M, \omega, H(a, b))$ is reversible with respect to Q, that is, there exists $Q : M \to M$ where Q is an anticanonical ($Q^*(\omega) = -\omega$) involution ($Q^{-1} = Q$) with $H \circ Q = H$. We call Q the reversibility of the system.

Theorem 1 *For Hamiltonian systems with a reversibility Q and a saddle-center equilibrium r, let χ be the set of fixed points of Q. If $r \in \chi$ and the unstable manifold of r intersects χ, then there exists an homoclinic orbit to r.*

PROOF.

From H2 and symplectic properties of ω one has:

$$\psi_t \circ Q = Q \circ \psi_{-t}. \tag{1}$$

Let ξ be a solution to the Hamiltonian system such that $\xi(0) \in \chi$. Equation (1) implies that $\xi(t) = \psi_t \circ \xi(0) = \psi_t \circ Q \circ \xi(0) = Q \circ \psi_{-t} \circ \xi(0) = Q \circ \xi(-t)$. Since $Q(r) = r$, if $\lim_{t \to -\infty} \xi(t) = r$ then $\lim_{t \to \infty} \xi(t) = \lim_{t \to -\infty} Q \circ \xi(t) = r$.

Thus the problem of finding an homoclinic orbit is replaced by the search of intersection of unstable orbits (in general one-dimensional) with the set χ (in general two- dimensional).

The majority of Hamiltonian systems that possess homoclinic orbits to saddle-center equilibrium points also exhibit some kind of discrete symmetry. In this case such an orbit is easily found by analysing some Hamiltonian sub-system with one degree of freedom. Unfortunately in several physically interesting systems this symmetry is unknown or non existent. Our work refers to homoclinic orbits in a class of systems with a kind of reversibility found in important physical problems and in differential equations. For such systems there is no symmetry that can be used to reduce the number of degrees of freedom and the method herein presented is a viable alternative to compute homoclinic orbits.

3 CONSTRUCTION OF HOMOCLINIC ORBITS IN MODEL SYSTEMS

We use the following fourth order family of differential equations as model systems

$$u^{iv} + au'' - u + f(u,b) = 0, \tag{2}$$

where $a \in \mathbb{R}$, $b \in \mathbb{R}$, f analytic. We transform 2 into an equivalent system

$$\begin{aligned} u' &= v \\ v' &= p_v \\ p_u' &= -u + f(u,b) \\ p_v' &= -p_u - av \end{aligned} \tag{3}$$

with Hamiltonian function $H(u, v, p_u, p_v, a, b) = p_u v + \frac{p_v^2}{2} + \frac{av^2}{2} + \frac{u^2}{2} - F(u, b)$, where F is a primitive of f.

Equation 2 turns up in several branches of physics, for instance, solitary waves in the presence of surface tension [7]. In this case f is approximated by u^2 and parameter a is related to the velocity of the wave. Other important cases in which equation 2 is relevant refers to localized patterns in elastic structures [9] and spatial patterns in phase transition [13] (in this context 2 is known as the Fisher-Kolmogorov stationary extended equation). For more applications of solitary waves see [15],[16] and [17].

Some authors ([18],[4],[19],[20] and [21]) have looked for the conditions on f that guarantee the existence of homoclinic orbits to $u = 0$, at least for some values of the parameters (a, b) and classes of such functions. However there has been no efforts, as yet, for members of such classes, to find the curves in parameter space where equation 2 presents homoclinic orbits to $u = 0$. In this context, our interest is in the following problem: given a family of functions f, find values (a, b) for which equation 2 has homoclinic solutions ϕ to the origin, that is $\lim_{t\to\pm\infty} \phi(t) = (0, 0, 0, 0)$. For one degree of freedom saddle-center hamiltonian system, see [22].

One can readily show that system 3 has a reversibility $Q : (u, v, p_u, p_v) \longmapsto (u, -v, -p_u, p_v)$, whose set of fixed points is $\chi = \{(u, v, p_u, p_v) \mid v = p_u = 0\}$. We define Φ as the set given by the intersection of the energy zero level, $\{H \equiv 0\}$, with a Poincaré section defined as $\{p_u = 0\}$. Here Φ is represented analytically by the equation $\frac{p_v^2}{2} + \frac{av^2}{2} + \frac{u^2}{2} - F(u, b) = 0$.

Given b, let $p(a)$ be the intersection of a solution of 3 with Φ. We expect that, given the continuous dependence of solutions with respect to the parameters, $p(a)$ intercepts the Poincaré section $\{v = 0\}$ inside Φ. In this way we determine for which parameters the respective solution intersects χ, the fixed points of the reversibility, a fact that characterizes the homoclinic orbit (figure 1). For more properties about homoclinic orbits and their relation with invariant sets, see [23],[24] and [25].

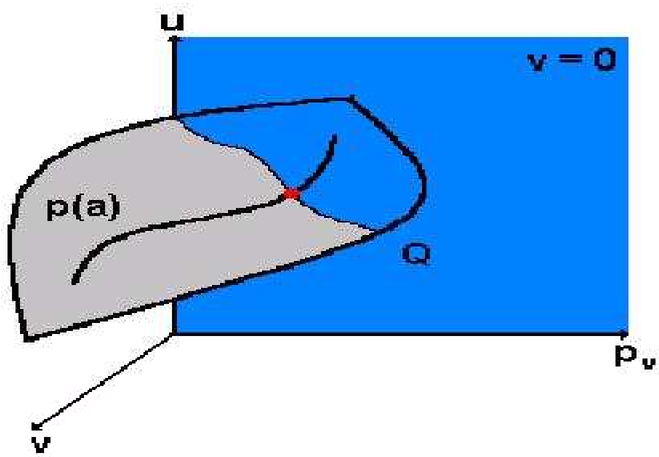

Fig. 1. Method Illustration

We illustrate these ideas using a specific equation in the family 2 with a known homoclinic orbit to $u = 0$, that is, $u(x) = sech(x)$. Thus

$u\'(x) = -sech^2(x)senh(x)$.

$u''(x) = -2sech(x)[-sech^2(x)senh(x)]senh(x) - sech^2(x)\cosh(x) =$

$= 2\underbrace{sech^3(x)}_{u^3}\underbrace{senh^2(x)}_{u^{-2}-1} - \underbrace{sech(x)}_{u}.$

Resulting in

$$u'' - u + 2u^3 = 0. \tag{4}$$

Multiplying 4 by $u\'$ and integrating we obtain the constant of motion

$$H(u, u\') = \frac{(u\')^2}{2} \underbrace{-\frac{u^2}{2} + \frac{u^4}{2}}_{V(u)} = E = cte. \tag{5}$$

Considering the level curves in Figure 2, we obtain a homoclinic orbit for $E = 0$.

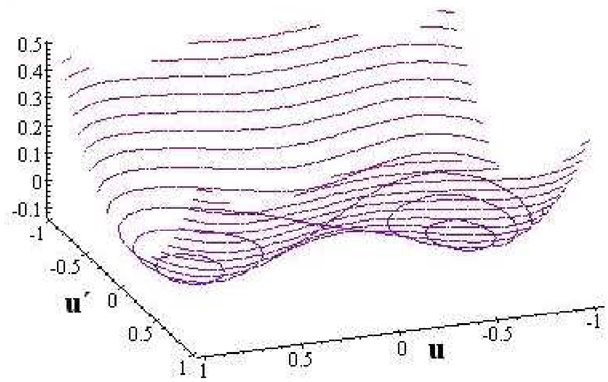

Fig. 2. Level Curves for 5

From 5 we plot the potential $V(u)$ in Figure 3 and get the critical point $u = 0$ for $E = 0$. Classically we obtain motion for $V(u) < E = 0$ since $\frac{(u')^2}{2} > 0$. This fact again guarantees the homoclinic property of the orbit.

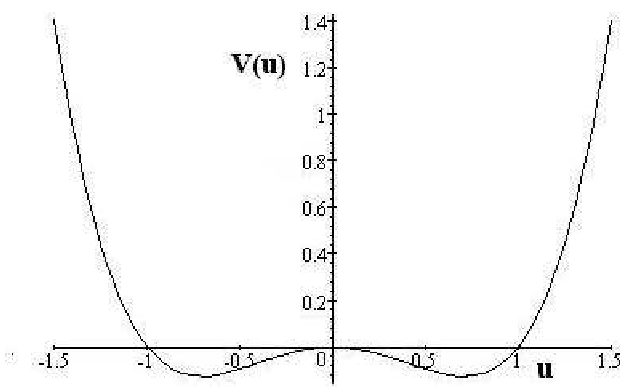

Fig. 3. Potential Function for 5

Taking the energy level $E = 0$ we derive, from 5,

$$(u')^2 - u^2 + u^4 = 0. \tag{6}$$

By 6, the second-order derivative of 4 and a change of coordinates lead to

$$u^{iv} + \frac{\sqrt{2}}{2}u'' - u + 11u^3 - 12u^5 = 0 \tag{7}$$

Or,

$$u^{iv} + au'' - u + f(u,b) = 0 \text{ with } f(u,b) = b(11u^3 - 12u^5) \tag{8}$$

which possesses an orbit Γ given by $u(x) = sech(x)$ homoclinic to $u = 0$ for the homoclinic values $(a,b) = \left(\frac{\sqrt{2}}{2}, 1\right)$.

For model 7, $\Phi = \left\{(u,v,p_u,p_v)\,|\,\frac{p_v^2}{2}+\frac{av^2}{2}+\frac{u^2}{2}-\left[b\left(\frac{11}{4}u^4-2u^6\right)\right]=0\right\}$ and we obtain energy zero surface for $(a,b) = \left(\frac{\sqrt{2}}{2},1\right)$ as shown in Figure 4. The structure of the corresponding set χ, which depends only on parameter b, is plotted in Figure 5.

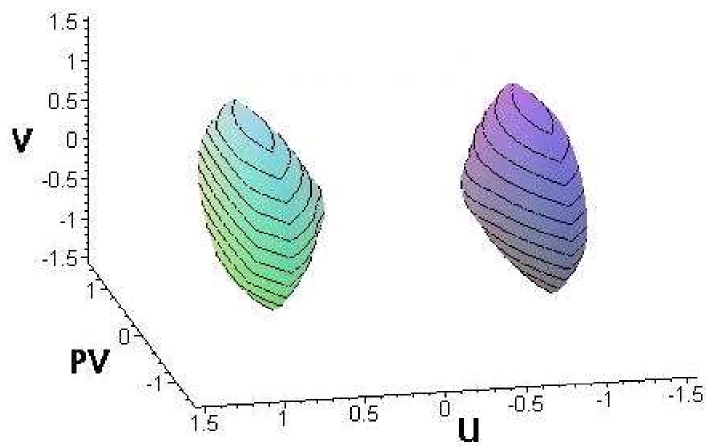

Fig. 4. 3D Energy Zero Surface

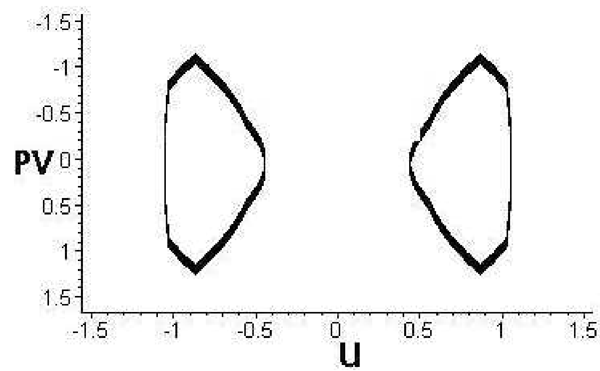

Fig. 5. χ set

Evolving system 8 towards the homoclinic point $(a,b) = \left(\frac{\sqrt{2}}{2},1\right)$, the orbit Γ in the unstable manifold hits the set χ of fixed points of the reversibility, "reverting" its behavior and connecting Γ to the stable manifold, thus characterizing the homoclinic orbit. For a perturbation of order 10^{-2} in parameter a there will be no intersections and the new orbit does not belong to the stable manifold of the equilibrium point, as shown in Figures 6 and 7.

Using the same procedure developed so far, we build the following equation from $u(x) = sech^2(x)$:

$$u^{iv}-\frac{15}{4}u''-u+3\left(\frac{65}{2}u^2-40u^3\right)=0. \tag{9}$$

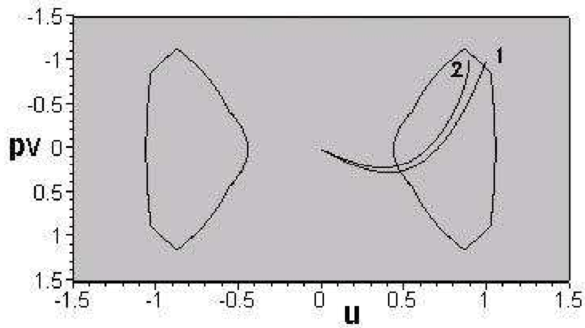

Fig. 6. 2D View of Homoclinic Orbit (1) and Small Pertubation (2)

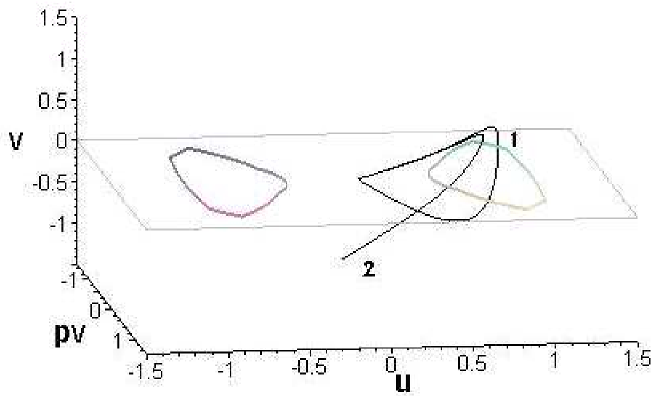

Fig. 7. 3D View of Homoclinic Orbit (1) and Small Pertubation (2)

We rewrite 9 as:

$$u^{iv} + au'' - u + f(u,b) = 0 \text{ where } f(u,b) = b\left(\frac{65}{2}u^2 - 40u^3\right) \qquad (10)$$

From this expression one can show that the orbit $\Gamma : u(x) = sech^2(x)$ is homoclinic to $u = 0$ for $(a,b) = \left(-\frac{15}{4}, 3\right)$

We developed an algorithm that runs through all points in a grid of (a,b) values, with spacing 10^{-2}, searching for intersections of orbits that belong to the instable manifold and the set of fixed points of the reversibility defined for the system 10. The result, as shown in figure 8, not only confirms the homoclinic value $(a,b) = \left(-\frac{15}{4}, 3\right)$ as expected, but also find an infinite number of additional values with this property.

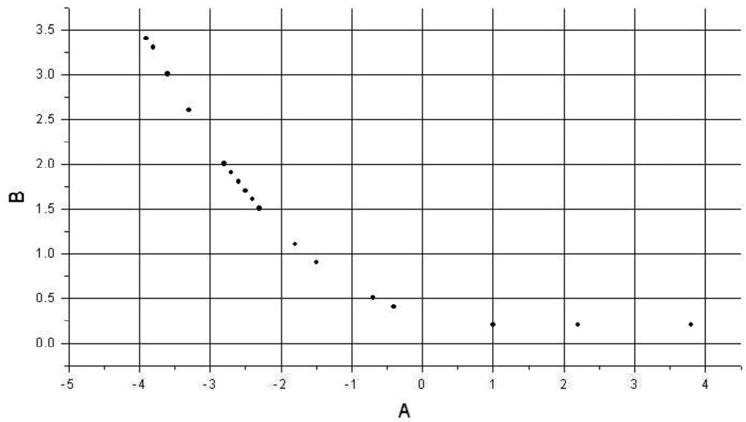

Fig. 8. Systems with Homoclinic Orbits in parameter space

4 Conclusions

The computational method developed in this work was based on geometrical and analytical properties observed in reversible hamiltonian systems. The main result contained in the theorem of section 2 does not restrict the hamiltonian dynamics. We have chosen saddle-center equilibriums in view of it relevance in many applications, but we could extend all results discussed in this work to a much wider variety of equilibriums. In [20] we find other examples of reversible hamiltonian systems and their applications.

Applying our method to model 2 with a known homoclinic orbit as seen in 7 and 10, we already had an indication of its efficiency. Not only the expected results were confirmed (figures 6 and 7) but an infinite set on new homoclinic values was found and plotted in figure 8, which graphic is similar to the bound states distribution in multi-pulse embedded solitons observed in [26]. This phenomenon is known to others authors as "cascade of homoclinic orbits" [3],[27] and as "explosion of chaotic sets" [24].

The next step is to apply the tool developed herein to others situations of interest. We are working with equation $u^{iv} + au'' - u + f(u, b) = 0$ where $f(u, b) = bu^2$; this system can be employed as a model of solitary waves in presence of superficial tension [7]. We intend to report in a forthcoming work the distribution of systems with homoclinic orbits in the space of parameters and the corresponding phase transition, as done in [26],[28] and [29], comparing our results with other analytical and experimental procedures. We hope to provide a "structural" point of view for systems with solitary waves in the set of hamiltonian vector fields with two degrees of freedom.

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